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LETTER TO THE EDITOR

Exact results for multiple state cellular automata

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Abstract. We present exact results for the dynamics of one-dimensional, probabilistic, multiple state cellular automata which map into generalised Potts models on anisotropic triangular lattices. We obtain analytic expressions for both the time evolution of their activity and the asymptotic behaviour of the correlation functions.

In recent years, cellular automata have attracted much interest since they seem to embody the time evolution of systems with many degrees of freedom (von Neumann 1966, Aladyeff 1974). The realisation that automata with binary states have a correspondence with the static properties of Ising-like systems has led to a number of studies which have improved our understanding of both sets of systems (Welberry and Galbraith 1973, 1975, Verhagen 1976, Welberry 1977, Enting 1977a, b, 1978a, b, Rujframework 1982, Domany and Kinzel 1983). Recently it has also been recognised that the equilibrium, static properties of a Potts model (Potts 1952, Wu 1982) can be mapped into the time evolution of a lower-dimensional automaton with multiple states (RIZ 1984, Domany 1984). This in turn raises questions about the dynamics of such automata, about which little is known.

This letter presents some exact results for the dynamics of one-dimensional multiple state cellular automata which correspond to Potts models in triangular lattices with anisotropic two-site interactions and three-site interaction for down pointing triangles. The approach used in this paper can be readily generalised to cover more complicated cases. Besides their intrinsic value, these results show that the master equation approach is of great use in treating the dynamics of complicated systems with many degrees of freedom.

Consider a chain of N cells each of which can have q possible states. We will represent the state of the cell on the ith site by the variable $\sigma_i = 1, 2, \ldots, q$ ($i = 1, 2, \ldots, N$). The state of the system can then be characterised by the N-tuple number $(\sigma_1, \sigma_2, \ldots, \sigma_N)$. It is assumed that at even time $t = 2n (n = 0, 1, 2, \ldots)$ odd indexed sites can change their states according to probabilities that depend only on the states of their neighbouring (even indexed) sites at time $t-1 = 2n-1$, while the even indexed sites do not change their states. Similarly, at odd times $t = 2n+1 (n = 0, 1, \ldots)$ only even indexed sites can change their states. Thus the time development of the whole system can be pictured by the space–time diagram shown in figure 1.

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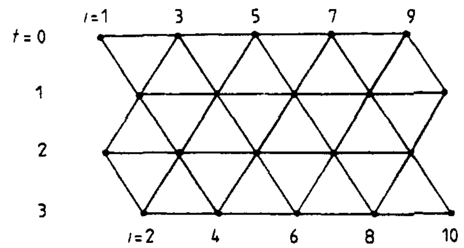

Figure 1. Space–time diagram of a one-dimensional cellular automaton.

To describe the time evolution of such a system we write down the master equation for $P(\alpha, t)$, the probability that the system is in the state $\alpha \equiv (\sigma_1, \sigma_2, \ldots, \sigma_N)$ at time t:

$$P(\alpha, t) - P(\alpha, t-1) = -\sum_\beta [p(\alpha \to \beta) P(\alpha, t-1) - p(\beta \to \alpha) P(\beta, t-1)],$$

where the transition probability $p(\alpha \to \beta)$ is the conditional probability for the whole system and which is given by the product of the individual conditional probabilities

$$p(\alpha \to \beta) \equiv p(\beta, t | \alpha, t-1) = \prod_i{}' p(\sigma_i' | \sigma_{i-1}, \sigma_{i+1}) \delta_{\sigma_{i-1}', \sigma_{i-1}} \tag{2}$$

with $\beta \equiv (\sigma_1', \sigma_2', \ldots, \sigma_N')$. The prime in the product implies the restriction $i + t = \text{odd}$.

To get an equation for $\langle \delta\mu\sigma_k \rangle \equiv \Sigma_{\{\sigma_i\}} \, \delta\mu\sigma_k P(\sigma_1, \sigma_2, \ldots, \sigma_N, t)$, which is just the probability that site k is in the state μ at time t, we multiply equation (1) by $\delta\mu\sigma_k$ and sum over α, i.e., sum over all σ_i's obtaining

$$\begin{aligned} \langle \delta\mu\sigma_k \rangle_t &= \langle \delta\mu\sigma_k \rangle_{t-1} \\ &= -\sum_\alpha \sum_\beta [\delta\mu\sigma_k p(\alpha \to \beta) P(\alpha, t-1) - \delta\mu\sigma_k p(\alpha \to \beta) P(\beta, t-1)] \\ &= -\left\langle \delta\mu\sigma_k \sum_\beta p(\alpha \to \beta) \right\rangle_{t-1} + \left\langle \sum_\beta \delta\mu\sigma_k' p(\alpha \to \beta) \right\rangle_{t-1}. \end{aligned} \tag{3}$$

If we now use equation (2) and note that $\Sigma_{\sigma_i'} p(\sigma_i' | \sigma_{i-1}, \sigma_{i+1}) = 1$, we obtain

$$\langle \delta\mu\sigma_k \rangle_t = \begin{cases} \langle \delta\mu\sigma_k \rangle_{t-1} & k + t = \text{even} \\ \langle p(\mu | \sigma_{k-1}, \sigma_{k+1} \rangle_{t-1} & k + t = \text{odd}. \end{cases} \tag{4}$$

Similarly, multiplying equation (1) by $\delta_{\mu\sigma_j} \delta_{\mu\sigma_k}$ ($|j - k| = \text{even}$) and summing over α gives the equation for $\langle \delta_{\mu\sigma_j} \delta_{\mu\sigma_k} \rangle_t$, i.e. the probability that sites j and k are in the same state μ at time t:

$$\langle \delta_{\mu\sigma_j} \delta_{\mu\sigma_k} \rangle_t = \begin{cases} \langle \delta_{\mu\sigma_j} \delta_{\mu\sigma_k} \rangle_{t-1} & j + t = \text{even} \\ \langle p(\mu | \sigma_{j-1}, \sigma_{j+1}) p(\mu | \sigma_{k-1}, \sigma_{k+1} \rangle_{t-1} & j + t = \text{odd}. \end{cases} \tag{5}$$

We now specify the conditional probabilities $p(\sigma_i' | \sigma_{i-1}, \sigma_{i+1})$ as follows

$$\begin{aligned} &p(\lambda | \mu\nu) = a, \qquad (\mu \neq \lambda, \nu \neq \lambda) \\ &p(\lambda | \lambda\mu) = a + b_1, \qquad p(\lambda | \mu\lambda) = a + b_2, \qquad p(\lambda | \lambda\lambda) = c. \end{aligned} \tag{6}$$

Furthermore the normalisation condition $\Sigma_{\sigma'_i} p(\sigma'_i|\sigma_{i-1}, \sigma_{i+1}) = 1$ gives the equality

$$(q-1)a + c = qa + b_1 + b_2 = 1 \tag{7}$$

which implies that only two of the four parameters (a, b_1, b_1, c) are independent. This specification corresponds to the simple form

$$p(\sigma'_i|\sigma_{i-1}, \sigma_{i+1}) = a + b_1\delta_{\sigma'_i\sigma_{i-1}} + b_2\delta_{\sigma'_i\sigma_{i+1}}. \tag{8}$$

Equation (8) allows us to write equations (4) and (5) in the explicit form

$$\langle\delta_{\mu\sigma_k}\rangle_t \equiv \chi^\mu_k(t) = \begin{cases} \chi^\mu_k(t-1) & k+t = \text{even} \\ a + b_1\chi^\mu_{k-1}(t-1) + b_2\chi^\mu_{k+1}(t-1) & k+t = \text{odd} \end{cases} \tag{9}$$

$$\langle\delta_{\mu\sigma_j}\delta_{\mu\sigma_k}\rangle_t \equiv R^\mu_{jk}(t) = \begin{cases} R^\mu_{jk}(t-1) & j+t = \text{even} \\ A^\mu_{jk}(t-1) + b_1^2 R^\mu_{j-1,k-1}(t-1) + b_2^2 R^\mu_{j+1,k+1}(t-1) & \\ \quad + b_1 b_2[R^\mu_{j-1,k+1}(t-1) + R^\mu_{j+1,k-1}(t-1)] & j+t = \text{odd} \end{cases} \tag{10}$$

where

$$A^\mu_{jk}(t) \equiv a^2 + ab_1[\chi^\mu_{j-1}(t) + \chi^\mu_{k-1}(t)] + ab_2[\chi^\mu_{j+1}(t) + \chi^\mu_{k+1}(t)]. \tag{11}$$

To solve equation (9), we consider the case $k + t = \text{odd}$ and write equation (9) in the form

$$\begin{aligned} \chi^\mu_k(t) &= a + b_1\chi^\mu_{k-1}(t-1) + b_2\chi^\mu_{k+1}(t-1) \\ &= a(1 + b_1 + b_2) + 2b_1 b_2\chi^\mu_k(t-2) + b_1^2\chi^\mu_{k-2}(t-2) + b_2^2\chi^\mu_{k+2}(t-2), \end{aligned} \tag{12}$$

where only odd or even indexed sites (but not both) are involved.

Introducing the generating function

$$F^\mu(\lambda, t) \equiv \sum_l{}' \lambda^l \chi^\mu_l(t) \tag{13}$$

where the prime implies the restriction $l + t = \text{odd}$, and λ can have any value for the case of infinite chains $(N \to \infty)$ (for the case of finite rings with periodic boundary condition, $\lambda^N = 1$), we get the equation for the generating function

$$\begin{aligned} F^\mu(\lambda, t) &= a(1 + b_1 + b_2)\sum_l{}' \lambda^l + (b_1\lambda + b_2\lambda^{-1})^2 F^\mu(\lambda, t-2) \\ &= a(1 + b_1 + b_2)\sum_l{}' \lambda^l \sum_{p=0}^{[t/2]-1} (b_1\lambda + b_2\lambda^{-1})^{2p} \\ &\quad + (b_1\lambda + b_2\lambda^{-1})^{2[t/2]} F^\mu(\lambda, t - 2[t/2]) \\ &= \sum_k{}' \lambda^k \{a(1 + b_1 + b_2)\sum_{p=0}^{[t/2]-1} (b_1 + b_2)^{2p} \\ &\quad + \sum_{p=0}^{2[t/2]} \binom{2[t/2]}{p} b_1^{2[t/2]-p} b_2^p \chi^\mu_{k-2[t/2]+2p}(t - 2[t/2]) \end{aligned}$$

which leads to the result $(k + t = \text{odd})$

$$\chi^\mu_k(t) = \frac{1 - (b_1 + b_2)^{2[t/2]}}{1 - (b_1 + b_2)} a + \sum_{p=0}^{2[t/2]} \binom{2[t/2]}{p} b_1^{2[t/2]-p} b_2^p \chi^\mu_{k-2[t/2]+2p}(t - 2[t/2]). \tag{15}$$

In the above equation, the first term is absent ($a=0$) if $b_1+b_2=1$. The final expression is then given by

$$\chi_k^\mu(t)=\begin{cases}\dfrac{1-(b_1+b_2)^t}{1-(b_1+b_2)}a+\displaystyle\sum_{p=0}^{t}\binom{t}{p}b_1^{t-p}b_2^p\chi_{k-t+2p}^\mu(0), & k=\text{odd}\\ \dfrac{1-(b_1+b_2)^t}{1-(b_1+b_2)}a+\displaystyle\sum_{p=0}^{t-1}\binom{t-1}{p}b_{b1}^{t-p-1}b_2^p[b_1\chi_{k-t+2p}^\mu(0)+b_2\chi_{k-t+2p+2(0)}^\mu], & k=\text{even}.\end{cases} \tag{16}$$

For $b_1+b_2<1$, this has the asymptotic behaviour

$$\chi_k^\mu(t)\xrightarrow{t\to\infty}\chi\equiv\frac{a}{1-(b_1+b_2)}=\frac{1}{q}, \tag{17}$$

which is the obvious result for the disordered state. We next solve equation (10). We will consider only the equilibrium solution, assuming that $R_{jk}^\mu(t)$ goes to the equilibrium value $R_{jk}^{(0)}$ in the asymptotic limit ($t\to\infty$). We try the form (keeping in mind that $|j-k|=\text{even}$),

$$R_{jk}^{(0)}=(\chi-\zeta)\eta^{|j-k|}+\zeta \tag{18}$$

which satisfies the condition

$$R_{kk}^{(0)}=\langle\delta_{\mu\sigma_k}\delta_{\mu\sigma_k}\rangle(0)=\langle\delta_{\mu\sigma_k}\rangle^{(0)}=\chi. \tag{19}$$

Upon substituting equation (20) into equation (11) in the asymptotic limit, we get

$$\zeta=\frac{A_{jk}^\mu(t\to\infty)}{1-(b_1+b_2)}=\left(\frac{a}{1-(b_1+b_2)}\right)^2=\chi^2$$

and

$$\eta=\frac{[1-(b_1-b_2)^2]^{1/2}-[1-(b_1+b_2)^2]^{1/2}}{2(b_1b_2)^{1/2}}\qquad(\leqslant 1) \tag{20}$$

which allows us to write the following expression for the correlation function,

$$\Gamma_{jk}^\mu\equiv R_{jk}^{(0)}-q^{-2}=(q-1)q^{-2}\eta^{|j-k|}. \tag{21}$$

As can be seen, it decays exponentially unless $\eta=1$ or $a=0$. Thus, the system has a phase transition only at the point $a=0$, $b_1+b_2=c=1$.

Finally, we consider the time retarded correlation function ($|j-k-t|=\text{even}$)

$$\begin{aligned}\langle\delta_{\mu\sigma_j}(t_0)\delta_{\mu\sigma_k}(t_0+t)\rangle&\equiv\sum_{\alpha,\beta}\delta_{\mu\sigma_k}p(\beta,t_0+t|\alpha,t_0)\delta_{\mu\sigma_j}P(\alpha,t_0)\\&=\sum_\alpha\langle\delta_{\mu\sigma_k}\rangle_{t+t_0}\delta_{\mu\sigma_j}P(\alpha,t_0)\end{aligned} \tag{22}$$

where it is understood that $\langle\delta_{\mu\sigma_k}\rangle_{t_0}=\delta\mu\sigma_k$. Equation (22) together with equations (16) and (21) leads to the desired expression, i.e.,

$$G^{\mu}_{jk}(t) \equiv \lim_{t_0\to\infty} \langle \delta_{\mu\sigma_j}(t_0)\delta_{\mu\sigma_k}(t_0+t)\rangle - q^{-2}$$

$$= \begin{cases} \dfrac{q-1}{q^2} \displaystyle\sum_{p=0}^{t} \binom{t}{p} b_1^{t-p} b_2^p \eta^{|j-k+t-2p|} & t=\text{even} \\ \dfrac{q-1}{q^2} \displaystyle\sum_{p=0}^{t-1} \binom{t-1}{p} b_1^{t-p-1} b_2^p (b_1\eta^{|j-k+t-2p|} + b_2\eta^{|j-k+t-2p-2|}), & t=\text{odd} \end{cases} \tag{23}$$

which has the asymptotic behaviour

$$G^{\mu}_{jk}(t) \xrightarrow{t\to\infty} \begin{cases} 0 & a\neq 0 \\ (q-1)q^{-2} & a=0. \end{cases} \tag{24}$$

The time development of one-dimensional multistate cellular automata, which we studied above, can be mapped into a two-dimensional Potts model (Rufán 1984, Domany 1984). If we regard the space–time lattice shown in figure 1 as a two-dimensional lattice, the probability of the whole two-dimensional configuration $\alpha \equiv \{\alpha_0, \alpha_1, \ldots, \alpha_M\}$ can be expressed as a product of conditional probabilities

$$P(\alpha) = p(\alpha_M, t=M|\alpha_{M-1}, t=M-1)\ldots p(\alpha_2, t=2|\alpha_1, t=1)p(\alpha_1, t=1|\alpha_0, t=0). \tag{25}$$

If we write the individual conditional probabilities $p(\sigma_i'|\sigma_{i-1}, \sigma_{i+1})$ in the form

$$p(\sigma_i'|\sigma_{i-1}, \sigma_{i+1}) = A\exp(K_1\delta\sigma_i'\sigma_{i-1} + K_2\delta\sigma_i'\delta_{i+1} + L\delta\sigma_i'\sigma_{i-1}\sigma_{i+1}) \tag{26}$$

where equation (6) gives the relation

$$A=a, \qquad e^{K_1} = 1+b_1a^{-1}, \qquad e^{K_2} = 1+b_2a^{-1}, \qquad e^L = ca/(a+b_1)(a+b_2) \tag{27}$$

then with equation (2) we get $P(\alpha)$ in the form familiar in equilibrium statistical mechanics

$$P(\alpha) \propto e^{-\beta H}$$

with the Hamiltonian

$$-\beta H = K_1 \sum_{\langle ij\rangle}{}_1 \delta_{\sigma_i\sigma_j} + K_2 \sum_{\langle ij\rangle}{}_2 \delta_{\sigma_i\sigma_j} + L \sum_{\langle ijk\rangle}{}_3 \delta_{\sigma_i\sigma_j\sigma_k}. \tag{28}$$

The first and second sums are to be done over two slanted bonds, respectively, and the third one is over down pointing triangles. Thus we get a Potts model in a triangular lattice with anisotropic two site interactions $(K_1, K_2, 0)$ and three-site interaction (L) for down pointing triangles. Equation (16) can be interpreted as giving the expectation value for a given boundary condition. Therefore we can solve for boundary explicitly. Within this framework equation (23) gives the correlation function between any two sites only if the boundary effect can be neglected. The critical point $a=0$ found previously corresponds to $T=0$ with ferromagnetic two-site interaction and antiferromagnetic three-site interaction, respectively.

This approach can be generalised to cover more complicated systems. Furthermore, we can specify the conditional probabilities more generally than in equation (6). In this case, there will arise more terms such as the external field in the corresponding Potts Hamiltonian other than those in equation (28). For example, we can make two channel problems in which both even and odd indexed sites can change their states at the same time. Also one can make the conditional probabilities depend on the states of themselves at previous time as well as those of neighbouring sites. In these cases, the corresponding Potts model will be different in either lattice structure or interaction type. Even in these more general cases, one can get some exact information for some subsets of the whole possible sets.

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